A note on *q*-multiple zeta functions

2001 J. Phys. A: Math. Gen. 34 L643

(http://iopscience.iop.org/0305-4470/34/46/101)

Institute of Physics Publishing Journal of Physics A: Mathematical and General
J. Phys. A: Math. Gen. **34** (2001) L643–L646 PII: S0305-4470(01)28824-6

LETTER TO THE EDITOR

A note on q-multiple zeta functions

Taekyun Kim

Institute of Science Education, Kongju National University, Kongju 314-701, Korea

E-mail: tkim@kongju.ac.kr and taekyun64@hanmail.net

Received 12 September 2001
Published 9 November 2001
Online at stacks.iop.org/JPhysA/34/L643

Abstract
The purpose of this Letter is to give the value of the q-multiple zeta function at negative integers, which is an answer to a part of the problem in a previous publication (Kim T, Park D-W and Rim S H 2001 *J. Phys. A: Math. Gen.* **34** 7633).

PACS number: 02.30.Gp

Mathematics Subject Classification: 11S80

1. Introduction

Let p be a fixed prime, and let $\mathbb{C}_p$ denote the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = \frac{1}{p}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then we normally assume $|q| < 1$, and when $q \in \mathbb{C}_p$, then we normally assume $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leqslant 1$. We use the notation

$$[x] = [x : q] = \frac{1-q^x}{1-q} = 1 + q + q^2 + \cdots + q^{x-1}.$$

In a recent paper (see [5]), we have considered the q-analogue of the multiple zeta function as follows. For $s \in \mathbb{C}$, $q \in \mathbb{C}$ with $|q| < 1$, define

$$\zeta_q^{(h,k)}(s) = \sum_{a_1,\ldots,a_k=0}^{\infty} \frac{q^{h(a_1+\cdots+a_k)}}{[a_1+\cdots+a_k]^s} + (q-1)\frac{1-s+h}{1-s} \sum_{a_1,\ldots,a_k=0}^{\infty} \frac{q^{h(a_1+\cdots+a_k)}}{[a_1+\cdots+a_k]^{s-1}} \tag{1}$$

where h, k are positive integers.

However, we could not find the analogues of Bernoulli numbers which $\zeta_q^{(h,k)}(s)$ can be viewed as interpolating at negative integers. This left this interpolation problem open, to find the analogues of Bernoulli numbers which $\zeta_q^{(h,k)}(s)$ can be viewed as interpolating at negative integers (see [5]). This problem is of some interest in connection with speculations about the new multiple zeta function associated with the quantization of a bosonic non-Archimedean-valued field to be carried out in the functional integral formalism (cf [1–4]).

In this Letter, we construct the analogues of Bernoulli numbers, which is an answer to a part of the above problem (cf [5]).

2. On the analogues of Bernoulli numbers

In this section, we assume $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. For d a fixed positive integer with $(p,d)=1$, let

$$X = X_d = \varprojlim_N \mathbb{Z}/dp^N\mathbb{Z} \qquad X_1 = \mathbb{Z}_p$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\,\mathbb{Z}_p$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\}$$

where $a \in \mathbb{Z}$ lies in $0 \leqslant a < dp^N$.

An invariant p-adic q-integral on $\mathbb{Z}_p$ of a uniformly differentiable function f was defined by

$$\int_{\mathbb{Z}_p} f(x)\,\mathrm{d}\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]} \sum_{0\leqslant j<p^N} f(j)q^j \qquad \text{(cf [5–8])}. \tag{2}$$

For $h, k \in \mathbb{N} = \{\text{the set of natural numbers}\}$, we consider the analogues of Bernoulli numbers by making use of p-adic q-integrals as follows:

$$\beta_m(h,k:q) = \underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k\ \text{times}} [x_1+x_2+\cdots+x_k]^m q^{(h-1)\sum_{i=0}^{k} x_i}\,\mathrm{d}\mu_q(x_1)\cdots\mathrm{d}\mu_q(x_k). \tag{3}$$

Note that $\lim_{q\to1}\beta(1,1:q) = B_m$, where B_m are the mth ordinary Bernoulli numbers (see [7]).

Theorem 1. *For $m \geqslant 0$, $h, k \in \mathbb{N}$, we have*

$$\beta_m(h,k:q) = \frac{1}{(1-q)^m}\sum_{j=0}^{m}\binom{m}{j}(-1)^j\left(\frac{j+h}{[j+h]}\right)^k. \tag{4}$$

Proof. We see

$$\left(\frac{1-q}{1-q^{p^N}}\right)^k \sum_{a_1=0}^{p^N-1}\cdots\sum_{a_k=0}^{p^N-1}[a_1+\cdots+a_k]^m q^{h(a_1+\cdots+a_k)}$$

$$= \left(\frac{1-q}{1-q^{p^N}}\right)^k \frac{1}{(1-q)^m}\sum_{a_1,\cdots a_k=0}^{p^N-1}\sum_{j=0}^{m}\binom{m}{j}(-1)^j q^{j\sum_{i=1}^k a_i} q^{h\sum_{i=1}^k a_i}$$

$$= \frac{1}{(1-q)^m}\sum_{j=0}^{m}\binom{m}{j}(-1)^j\left(\frac{1-q}{1-q^{p^N}}\right)^k \underbrace{\frac{1-q^{(j+h)p^N}}{1-q^{j+h}}\cdots\frac{1-q^{(j+h)p^N}}{1-q^{j+h}}}_{k\ \text{times}}.$$

Since $\lim_{n\to\infty} q^{p^n} = 1$ for $|1-q|_p < 1$, our assertion follows. □

Let $G^{(h,k)}(t)$ be the generating function of $\beta(h,k:q)$ as follows:

$$G^{(h,k)}(t) = \sum_{n=0}^{\infty}\beta_n(h,k:q)\frac{t^n}{n!}. \tag{5}$$

Thus we have

$$\begin{aligned} G^{(h,k)}(t) &= \sum_{l=0}^{\infty}\left(\frac{1}{(1-q)^l}\sum_{i=0}^{l}\binom{l}{i}(-1)^i\left(\frac{i+h}{[i+h]}\right)^k\right)\frac{t^l}{l!} \\ &= \sum_{j=0}^{\infty}\left(\frac{j+h}{[j+h]}\right)^k\frac{(-1)^j}{(1-q)^j}\frac{t^j}{j!}\sum_{i=0}^{\infty}\left(\frac{1}{1-q}\right)^i\frac{t^i}{i!} \\ &= \mathrm{e}^{\frac{t}{1-q}}\sum_{j=0}^{\infty}\left(\frac{j+h}{[j+h]}\right)^k\frac{(-1)^j}{(1-q)^j}\frac{t^j}{j!}. \end{aligned} \tag{6}$$

Note that

$$q^h G^{(h,1)}(qt)\mathrm{e}^t - t = G^{(h,1)}(t). \tag{7}$$

By (5) and (7), we have

$$q^h(q\beta(h,1:q)+1)^m - \beta_m(h,1:q) = \begin{cases} 1 & \text{if}\quad m=1 \\ 0 & \text{if}\quad m>1 \end{cases} \tag{8}$$

where we use the usual convention about replacing $\beta^i(h,1:q)$ by $\beta_i(h,1:q)$ $(i \geqslant 0)$.

3. q-multiple zeta functions

In this section, we assume $q \in \mathbb{C}$ with $|q| < 1$. To give the analogues of Bernoulli numbers which $\zeta_q^{(h,k)}(s)$ can be viewed as interpolating at negative integers, we need to modify the numbers $\beta_m(h,k:q)$ as follows:

$$B_m(h,k:q) = \frac{1}{(1-q)^{m+k-1}}\sum_{j=0}^{m}\binom{m}{j}(-1)^j\frac{j+h}{[j+h]^k}. \tag{9}$$

It is easy to see that $\beta_m(h,1:q) = B_m(h,1:q)$.

Let $F^{(h,k)}(t)$ be the generating function of $B_m(h,k:q)$:

$$F^{(h,k)}(t) = \sum_{n=0}^{\infty} B_n(h,k:q)\frac{t^n}{n!}.$$

By the same method as (6), we easily see

$$F^{(h,k)}(t) = \frac{1}{(1-q)^{k-1}}\left(\sum_{i=0}^{\infty}\frac{i+h}{[i+h]^k}\left(\frac{1}{q-1}\right)^i\frac{t^i}{i!}\right)\mathrm{e}^{\frac{t}{1-q}}. \tag{10}$$

Thus we have

$$\begin{aligned} F^{(h,k)}(t) &= \sum_{m=0}^{\infty}\left(\frac{1}{(1-q)^{m+k-1}}\sum_{j=0}^{m}\binom{m}{j}(-1)^j\frac{j+h}{[j+h]^k}\right)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\Bigg(-m\sum_{a_1,\ldots,a_k=0}^{\infty} q^{h\sum_{i=1}^{k}a_i}\left[\sum_{i=1}^{k}a_i\right]^{m-1} \\ &\qquad -(q-1)(m+h)\sum_{a_1,\ldots,a_k=0}^{\infty} q^{h\sum_{i=1}^{k}a_i}\left[\sum_{i=1}^{k}a_i\right]^m\Bigg)\frac{t^m}{m!} \end{aligned} \tag{11}$$

where $\left[\sum_{i=1}^{k}a_i\right]^m = [a_1+a_2+\cdots+a_k]^m$.

Differentiating both sides with respect to t in (11) and comparing coefficients, we obtain the following:

Theorem 2. *For $m \geqslant 0$, $h, k \in \mathbb{N}$, we have*

$$B_m(h, k : q) = -m \sum_{a_1,\ldots,a_k=0}^{\infty} q^{h\sum_{i=1}^{k} a_i} \left[\sum_{i=1}^{k} a_i\right]^{m-1} - (q-1)(m+h) \times \sum_{a_1,\ldots,a_k=0}^{\infty} q^{h\sum_{i=1}^{k} a_i} \left[\sum_{i=1}^{k} a_i\right]^{m}$$

that is

$$-\frac{B_m(h, k : q)}{m} = \sum_{a_1,\ldots,a_k=0}^{\infty} q^{h\sum_{i=1}^{k} a_i} \left[\sum_{i=1}^{k} a_i\right]^{m-1} + (q-1)\frac{(m+h)}{m} \times \sum_{a_1,\ldots,a_k=0}^{\infty} q^{h\sum_{i=1}^{k} a_i} \left[\sum_{i=1}^{k} a_i\right]^{m}.$$

By theorem 2, note that

$$\zeta_q^{(h,k)}(1-m) = -\frac{B_m(h, k : q)}{m} \qquad \text{for} \quad m \geqslant 1$$

which is an answer to a part of the problem in [5].

This work was supported by a Korean Research Foundation grant (KRF-99-005-D00026).

References

[1] Albeverio S, Bayod J M, Perez-Garcia C, Cianci R and Khrennikov A Yu 1999 *Acta Appl. Math.* **57** 205

[2] Dubischar D, Gundlach V M, Steinkamp O and Khrennikov A Yu 1999 *Nuovo Cimento* B **114** 373

[3] Khrennikov A Yu 1994 *p-adic Valued Distributions in Mathematical Physics* (Dordrecht: Kluwer)

[4] Khrennikov A Yu 1997 *Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models* (Dordrecht: Kluwer)

[5] Kim T, Park D-W and Rim S H 2001 *J. Phys. A: Math. Gen.* **34** 7633

[6] Kim T 1998 Multiple zeta values, di-zeta values and their applications *Lecture Notes in Number Theory (Kyungnam University)* (Gu-Duk)

[7] Kim T and Rim S H 2001 *Notes on Number Theory and Discrete Math.* vol 7 (Sofia: Academic)

[8] Murphy G J 2001 *Advanced Studies in Contemporary Mathematics* vol 3 (Pusan: Ku-Deok) p 31